Informational Approach to Identical Particles in Quantum Theory

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A remarkable feature of quantum theory is that particles with identical intrinsic properties must be treated as indistinguishable if the theory is to give valid predictions. For example, our understanding of the structure of the periodic table hinges on treating the electrons in multi-electron atoms as indistinguishable. In the quantum formalism, indistinguishability is expressed via the symmetrization postulate[1,2], which restricts a system of identical particles to the set of symmetric states ('bosons') or the set of antisymmetric states ('fermions'). However, the precise connection between particle indistinguishability and the symmetrization postulate has not been established. There exist a number of variants of the postulate that appear to be compatible with particle indistinguishability[3,4], and a well-known derivation of the postulate implies that its validity depends on the dimensionality of space[4,5]. These variants leave open the possibility that there exist elementary particles, such as anyons[4,6], which violate the symmetrization postulate. Here we show that the symmetrization postulate can be derived on the basis of the *indistinguishability postulate.* This postulate establishes a functional relationship between the amplitude of a process involving indistinguishable particles and the amplitudes of all possible transitions when the particles are treated as distinguishable. The symmetrization postulate follows by requiring consistency with the rest of the quantum formalism. The key to the derivation is a strictly informational treatment of indistinguishability which prohibits the labelling of particles that cannot be experimentally distinguished from one another. The derivation implies that the symmetrization postulate admits no natural variants. In particular, the existence of anyons as elementary particles is excluded.

If two particles—two electrons, for example—are identical in all their intrinsic properties such as mass and charge, they cannot be distinguished from one another by measurements of these properties. In classical physics, however, one can in principle follow the precise trajectory of each particle, and thereby consistently label each particle over time. For this reason, identical particles in classical physics are generally regarded as distinguishable. One of the most remarkable features of quantum theory is that, if the theory is to give predictions in accordance with experiment, identical particles must be treated as *indistinguishable*. For example, our understanding of the structure of the periodic table hinges on the assumption that the electrons in multi-electron atoms are indistinguishable. Formally, the indistinguishability of the particles in a quantum system is expressed via the *symmetrization postulate*. The postulate restricts the system to the set of symmetric states or the set of antisymmetric states, and is responsible for the classification of all identical particles in nature into bosons (symmetric states) and fermions (antisymmetric states). The postulate was proposed in 1926 as a way of incorporating Pauli's exclusion principle into the nascent formalism of quantum theory[1,2], and has since proven essential to our understanding of a vast range of physical phenomena.

Given its importance and formal simplicity, one would hope for a direct path from the assumption that identical particles are indistinguishable to the symmetrization postulate itself. However, such a path has proved elusive, leading to the suggestion that there may exist various elementary particles in nature, in addition to fermions and bosons, which violate the symmetrization postulate. For example, although Dirac asserted that the restriction to states that are symmetric or antisymmetric follows from the theoretical necessity of eliminating a formal redundancy[1,7], many other restrictions are in fact mathematically possible[3]. These possibilities suggest that non-standard elementary particles, called *paraparticles,* may exist in nature. Despite investigation[8,9], paraparticles have yet to be ruled out experimentally or theoretically. Many attempts have been made to derive the symmetrization postulate from physical principles[4,5,10–17], but the results are generally incomplete or inconclusive. For example, the well-known topological approach[4,5] yields the symmetrization postulate in the special case of structureless particles moving in three or more spatial dimensions, but does not yield a general statement applicable to any system of identical subsystems. The approach also implies the breakdown of the symmetrization postulate in two spatial dimensions, leaving open the possibility that there exist non-standard elementary particles—so-called *anyons*—in two dimensions[4,6]. Although anyons have found significant application[18,19], for example in the explanation of the fractional Hall effect, their status as *elementary* particles on a par with fermions and bosons is uncertain insofar as the topological approach relies upon abstract assumptions, in particular upon a conjectured generalization[20] of Feynman's sum rule for multiply-connected spaces.

Here we show that the symmetrization postulate can be derived in full generality from a novel postulate, the *indistinguishability postulate*. The postulate establishes a relation between the theoretical description of two different experiments, positing that the amplitude of a process involving several indistinguishable subsystems (hereafter referred to as *particles*) is determined by the amplitudes of all possible transitions of these particles when treated as distinguishable. The derivation harnesses the frame-

work and methodology recently used to derive Feynman's formalism of quantum theory[21], in three key ways. First, our treatment of indistinguishability is fully informational: if two particles are indistinguishable on the basis of information obtainable through measurements on them, we refrain from labelling the particles. In contrast, previous approaches[1,4,5,10,11,14,15] typically label indistinguishable particles, and then require that predictions are insensitive to relabelling. As we shall see, our informational approach allows us to clearly grasp the essential meaning of the symmetrization postulate, and to steer clear of conceptual pitfalls that commonly afflict treatments of this subject[22–24]. Second, we make use of the insight that Feynman's rules are best regarded as establishing a formal relationship between different experiments[21,25], an insight that lies at the basis of the indistinguishability postulate. Third, we employ the simple yet powerful[21,25] requirement of consistency: if it is possible to compute an amplitude in two different ways, they must agree.

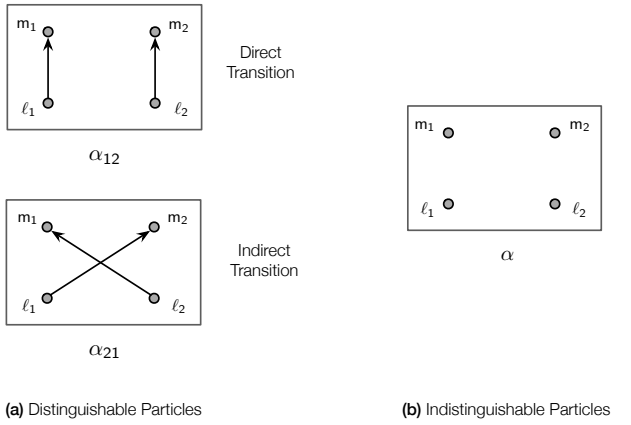

FIG. 1: *Measurements on two particles, distinguishable and indistinguishable.* Two particles are each subject to measurements at times, t_1 and t_2, yielding outcomes ℓ_1, ℓ_2 at t_1 and m_1, m_2 at t_2. *Left:* If the particles are distinguishable, one can experimentally determine which transition actually occurs—the 'direct' one (*top*), or the 'indirect' one (*bottom*). *Right:* If the particles are indistinguishable, one cannot say what happened in the intermediate time on the basis of the information obtained.

Two Indistinguishable Particles

Consider a quantum system of two particles, each subject to measurements (say, of position) at successive times t_1 and t_2. Suppose that the two measurements performed at t_1 yield outcomes ℓ_1 and ℓ_2, and that the measurements at t_2 yield outcomes m_1 and m_2. If it is possible to distinguish between the two particles, then there is some measurement (for example, of mass or charge) that we can, in principle, perform on each particle immediately before t_1 and after t_2 that allows us to conclude, for instance, that the particle responsible for outcome ℓ_1 is the same as that responsible for m_1 (see Fig. 1). If, however, the particles cannot be distinguished from one another, we cannot draw such a conclusion. Even the conclusion that "the particle that generated outcome ℓ_1 was also responsible for outcome m_1 or m_2, but we do not know which" is invalid since we cannot verify this statement on account of particle indistinguishability. To nonetheless draw such a conclusion would be tantamount to assuming the separate and continued existence of the particles between observations, which assumption our experience with quantum phenomena should warn us against. In particular, in the case of electron diffraction at a double-slit[25,26], in the absence of which-way detectors at each slit, we cannot infer from the experimental data that the electron passed through one slit or the other; if we nonetheless insist on drawing this inference, our predictions are incorrect—we are unable to account for the observed diffraction pattern.

As we have no way of 'looking inside' the indistinguishable-particle process, we must treat it as a black box. As a result, we cannot directly compute the amplitude, α, of the indistinguishable-particle process. Again, electron diffraction is analogous: the transition amplitude of an electron from source to screen in the absence of which-way detectors cannot be computed directly. Instead, it is obtained (via Feynman's amplitude sum rule) from the sum of the amplitudes of the two possible transitions through the slits with the detectors present, which amplitudes *can* be calculated[25]. In the present case of two particles, then, we seek to relate the amplitude of the indistinguishable-particle transition, α, with the amplitudes, α_{12} and α_{21}, of the 'direct' and 'indirect' transitions of the two distinguishable particles, where each of these particles agree in all their dynamically-relevant properties with the indistinguishable particles. Now, we cannot relate α to α_{12} and α_{21} via the sum rule since the sum rule only applies when different experiments are applied to the *same* system, whereas α refers to indistinguishable particles, while α_{12} and α_{21} refer to distinguishable particles. So, in order to derive a connection between α and α_{12} and α_{21}, we must first assume that, as in the case of electron diffraction, such a relation exists. Accordingly, we postulate that

$$\alpha = H(\alpha_{12}, \alpha_{21}), \tag{1}$$

where H is some continuous function to be determined (see Fig. 2(a)). Similarly, when two particles are subject to three successive measurements, we postulate that the indistinguishable-particle amplitude, γ, is given by

$$\gamma = G(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}), \tag{2}$$

where $\gamma_{11}, \gamma_{12}, \gamma_{21}$, and γ_{22} are the amplitudes of the corresponding distinguishable-particle transitions (see Fig. 2(b)), and G is a function to be determined. Both of these postulates are special cases of the indistinguishability postulate stated above.

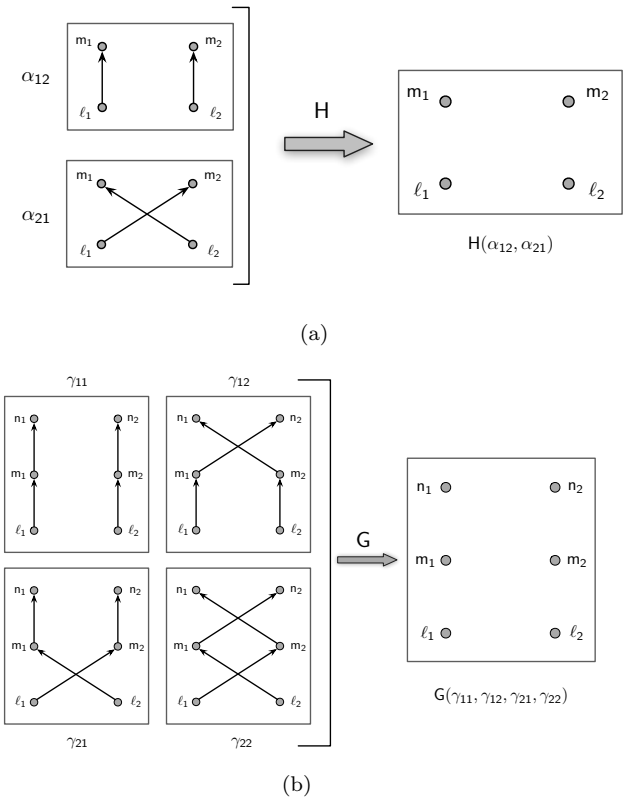

FIG. 2: *Amplitude for one- and two-stage experiments on two indistinguishable particles.* Two indistinguishable particles are each subject to a measurement at successive times. (a) Measurements at time t_1 and t_2 yield outcomes ℓ_1, ℓ_2 and m_1, m_2, respectively. The figures on the left show the transitions of two *distinguishable* particles compatible with these outcomes: the 'direct' transition of amplitude α_{12}, and the 'indirect' transition of amplitude α_{21}. We postulate that the amplitude of the indistinguishable-particle process is $H(\alpha_{12}, \alpha_{21})$, where H is a continuous function to be determined. (b) Measurements are performed at three successive times, yielding the indicated outcomes. On the left are shown the four possible transitions of two distinguishable particles, with respective amplitudes $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$, compatible with the observed outcomes. We postulate that the amplitude of sequence C is given by $G(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22})$.

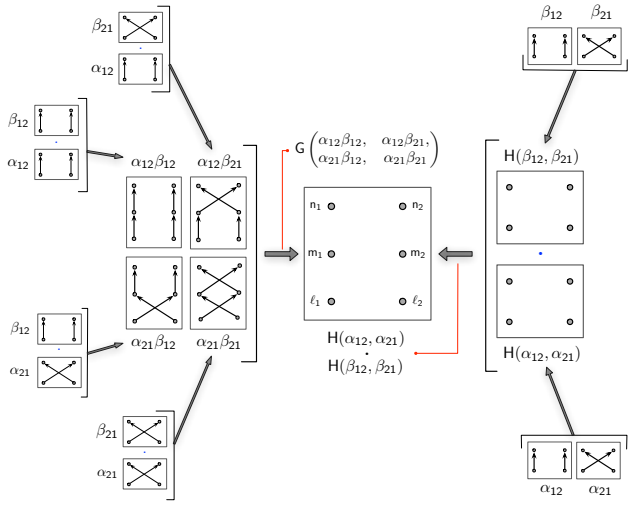

FIG. 3: *Origin of the G-product equation.* In the center, two indistinguishable particles are each subject to measurements at times t_1, t_2 and t_3, yielding the indicated outcomes. The amplitude for this process can be computed in two different ways. On the left are shown the four possible transitions of two distinguishable particles compatible with the observed sequence. These transitions can each be expressed as series $(\cdot)$ combinations of the indicated transitions, yielding amplitudes $\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}$. Hence, from Eq. (2)), the indistinguishable-particle process amplitude is $G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21})$. On the right, the indistinguishable-particle process is first decomposed into two smaller indistinguishable-particle processes in series $(\cdot)$, with respective amplitudes, $H(\alpha_{12}, \alpha_{21})$ and $H(\beta_{12}, \beta_{21})$, yielding an overall indistinguishable-particle amplitude $H(\alpha_{12}, \alpha_{21})\, H(\beta_{12}, \beta_{21})$.

The two postulates just given determine the symmetrization postulate for two particles up to an overall multiplicative factor. To fix this factor, we make use of the following. We note that, when $\alpha_{21} = 0$ but $\alpha_{12} \neq 0$, we have sufficient information to conclude that the particle detected at m_1 is the one that was earlier found at ℓ_1. Hence, in this case, the particles are distinguishable, and the usual rules for computing transitional probabilities must apply. Since the direct transition, with amplitude α_{12}, is the only one possible, the corresponding probability $|\alpha_{12}|^2$ must agree with the probability $|H(\alpha_{12}, 0)|^2$ computed using H. Hence,

$$|H(\alpha_{12}, 0)| = |\alpha_{12}|. \tag{3}$$

We shall now show that these assumptions determine the precise form of symmetrization postulate for two particles. The general case of N particles, together with additional detail and background terminology, can be found in Appendix A.

To determine G and H, we repeatedly use idea that, if it is possible to compute an amplitude in two different ways using Feynman's rules for single systems[21,27] (see also Appendix A 2 a) and the assumptions given above, consistency requires that they be equal. Each such call for consistency yields a functional equation. By considering a few special cases, we are able to formulate a set of functional equations that determine G and H.

First, consider a two-stage experiment in which the intermediate outcomes m_1 and m_2 are both atomic[21] (see Appendix A for terminology). We can derive the indistinguishable-particle amplitude, γ, in two different ways (see Fig. 3). Let α_{12}, α_{21}, respectively, be the amplitudes of the direct and indirect transitions in the first

stage, and β_{12}, β_{21} the corresponding amplitudes in the second stage. Using the amplitude product rule, we can directly compute the values $\gamma_{11} = \alpha_{12}\beta_{12}$, $\gamma_{12} = \alpha_{12}\beta_{21}$, $\gamma_{21} = \alpha_{21}\beta_{12}$, and $\gamma_{22} = \alpha_{21}\beta_{21}$. Then, using the G-function defined above,

$$\gamma = G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}).$$

Alternatively, we can directly apply the amplitude product rule to the system of identical particles to obtain

$$\gamma = H(\alpha_{12}, \alpha_{21})\, H(\beta_{12}, \beta_{21}).$$

Equating these two expressions, we obtain a functional equation, the G-product equation,

$$G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}) = H(\alpha_{12}, \alpha_{21})\, H(\beta_{12}, \beta_{21}). \quad (4)$$

Next, consider a two-stage experiment in which the outcome of one of the measurements performed at t_2 is a coarse-graining of atomic outcomes m_2 and m_2', which outcome we denote (m_2, m_2') (see Fig. 5). Here, α_{12}, α_{21} are respectively the first-stage amplitudes of the direct and indirect transitions compatible with the indistinguishable-particle process through m_2; $\alpha_{12}', \alpha_{21}'$ are similarly the amplitudes corresponding to the process through m_2'; and β_{12}, β_{21} are the second-stage amplitudes of the direct and indirect transitions compatible with the indistinguishable-particle process through both m_2 and m_2'.

The indistinguishable-particle amplitude, $\tilde{\gamma}$, in this case can be computed in two different ways. First, using the amplitude sum and product rules, we can write

$$\tilde{\gamma} = H(\alpha_{12}, \alpha_{21})\, H(\beta_{12}, \beta_{21}) + H(\alpha_{12}', \alpha_{21}')\, H(\beta_{12}, \beta_{21}),$$

Alternatively, one can compute $\tilde{\gamma}$ directly from Eq. (2):

$$\tilde{\gamma} = G\begin{pmatrix} (\alpha_{12}+\alpha_{12}')\beta_{12}, (\alpha_{12}+\alpha_{12}')\beta_{21}, \\ (\alpha_{21}+\alpha_{21}')\beta_{12}, (\alpha_{21}+\alpha_{21}')\beta_{21} \end{pmatrix},$$

where we have used the amplitude sum rule to compute the amplitude of the four transitions compatible with the indistinguishable-particle process. Hence, we have the functional equation

$$G\begin{pmatrix} (\alpha_{12}+\alpha_{12}')\beta_{12}, (\alpha_{12}+\alpha_{12}')\beta_{21}, \\ (\alpha_{21}+\alpha_{21}')\beta_{12}, (\alpha_{21}+\alpha_{21}')\beta_{21} \end{pmatrix} = H(\alpha_{12}, \alpha_{21})\, H(\beta_{12}, \beta_{21}) + H(\alpha_{12}', \alpha_{21}')\, H(\beta_{12}, \beta_{21}). \quad (5)$$

Finally, we consider the effect of swapping the labels on the outcomes m_1 and m_2 in an experiment (see Fig. 4). Since the particles are indistinguishable, there is no physical distinction between these two outcomes. Therefore, consistency requires that the predicted transition probabilities do not depend upon this labelling. As shown in the figure, one obtains $H(\alpha_{12}, \alpha_{21})$ and $H(\alpha_{21}, \alpha_{12})$ in the original and relabelled case, respectively. Although

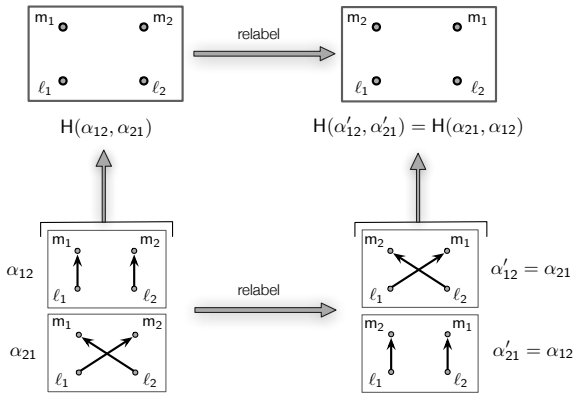

FIG. 4: *Relabelling Invariance.* A system of two indistinguishable particles are subject to measurements at two successive times t_1 and t_2. The experimental situations shown on the top-left and top-right are the same, except that the labels on the outcomes at t_2 are swapped. Accordingly, as shown, one can compute the amplitude of the indistinguishable-particle transition in two distinct ways, namely as $H(\alpha_{12}, \alpha_{21})$ or as $H(\alpha_{12}', \alpha_{21}') = H(\alpha_{21}, \alpha_{12})$. Since the actual experimental situation is the same in the two cases, these two expressions must yield the same transition probabilities. Hence, $|H(\alpha_{12}, \alpha_{21})| = |H(\alpha_{21}, \alpha_{12})|$.

these amplitudes may differ, the corresponding probabilities cannot, which implies

$$|H(\alpha_{12}, \alpha_{21})| = |H(\alpha_{21}, \alpha_{12})|. \quad (6)$$

We show in the Methods Summary that the functional equations (4), (5) and (6), together with Eq. (3), imply, without loss of generality, that

$$H(\alpha_{12}, \alpha_{21}) = \alpha_{12} \pm \alpha_{21}. \quad (7)$$

The $\pm$ sign is the only remaining degree of freedom, corresponding to bosons and fermions. This completes the derivation of the symmetrization postulate for two identical, indistinguishable particles as expressed[26] in Feynman's formulation of quantum theory.

The derivation for N particles given in Appendix A follows a similar line of argument, and yields the result

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) = \sum_{\boldsymbol{\pi} \in S_N} (\mathrm{sgn}(\boldsymbol{\pi}))^{\sigma}\ \alpha_{\boldsymbol{\pi}}, \quad (8)$$

where $\sigma = 0$ or $\sigma = 1$ is the only remaining degree of freedom.

Conclusion

We have shown that there exists a direct path from the assumption that identical particles are indistinguishable

to the symmetrization postulate. The result applies to any quantum system, such as system of identical particles with spin, or a system of identical abstract finite dimensional subsystems, and relies on the standard quantum formalism rather than any variant thereof. The derivation is a natural extension and application of the ideas previously used to derive Feynman's rules of quantum theory. This demonstrates that the symmetrization postulate is comparable in nature and reliability to the core quantum formalism, contrary to what has previously appeared to be the case. From the point of view of the derivation, the symmetrization postulate admits no natural variants, and, *contra* the topological approach, is unrelated to the dimension of space. In particular, the derivation implies that anyons do not exist as elementary particles on a par with fermions and bosons, and so must be viewed as composite entities[6].

As detailed elsewhere[25], the methodology used to derive Feynman's rules has been previously used to derive[28,29] the formalism of probability theory starting from Boolean logic, and has recently been applied to yield important insights in other domains[29]. The successful application of this methodology to the symmetrization postulate leads us to anticipate that this methodology may provide a valuable tool in understanding and constructing physical theory.

Methods Summary

Using Eq. (4), the left-hand side of Eq. (5) can be rewritten to give

$$H(\alpha_{12}+\alpha'_{12},\alpha_{21}+\alpha'_{21})\,H(\beta_{12},\beta_{21}) = H(\alpha_{12},\alpha_{21})\,H(\beta_{12},\beta_{21}) + H(\alpha'_{12},\alpha'_{21})\,H(\beta_{12},\beta_{21}).$$

A nontrivial solution requires that $H(\beta_{12},\beta_{21})$ cannot be zero for all β_{12},β_{21}. Hence, we obtain the additivity relation

$$H(\alpha_{12}+\alpha'_{12},\alpha_{21}+\alpha'_{21}) = H(\alpha_{12},\alpha_{21})+H(\alpha'_{12},\alpha'_{21}). \quad (9)$$

We are now in a position to determine H. First, Eq. (9) implies

$$H(\alpha_{12},\alpha'_{21}) = H(\alpha_{12}+0,0+\alpha'_{21}) = H(\alpha_{12},0)+H(0,\alpha'_{21}), \quad (10)$$

and Eq. (4) implies that

$$H(z_1z_2,0)\,H(1,0) = H(z_1,0)\,H(z_2,0). \quad (11)$$

If $H(1,0)=0$, the above equation gives that $H(z,0)=0$ for all z, which, using Eqs. (6) and (10), implies $H(z_1,z_2)=0$ for all z_1,z_2, the trivial solution. Therefore, $H(1,0)\neq 0$.

Now, Eq. (4) implies that

$$G(0,z,0,0) = H(z,0)\,H(0,1) = H(1,0)\,H(0,z), \quad (12)$$

which we can use to rewrite Eq. (10) as

$$H(\alpha_{12},\alpha_{21}) = H(\alpha_{12},0) + \frac{H(0,1)}{H(1,0)}H(\alpha_{21},0). \quad (13)$$

To determine the form of $H(z,0)$, we note from Eq. (9) that

$$H(z_1+z_2,0) = H(z_1,0)+H(z_2,0). \quad (14)$$

Writing $f(z)=H(z,0)/H(1,0)$, Eqs. (11) and (14) can be written as a pair of functional equations,

$$f(z_1+z_2) = f(z_1)+f(z_2) \quad (15a)$$
$$f(z_1z_2) = f(z_1)\,f(z_2), \quad (15b)$$

where $f(\cdot)$ is a complex-valued function of a complex argument. The continuous solutions of these equations in the domain $|z|\leq 1$ are $f(z)=z$, $f(z)=z^*$ or $f(z)=0$ (see Appendix B). The zero solution is inadmissible since, from its definition, $f(1)=1$. Therefore,

$$H(z,0) = \begin{cases} H(1,0)\,z \\ H(1,0)\,z^*. \end{cases}$$

From Eq. (6), we find $|H(1,0)| = |H(0,1)|$, so that $H(0,1)/H(1,0)=e^{i\theta}$. Therefore, Eq. (13) becomes

$$H(\alpha_{12},\alpha_{21}) = \begin{cases} H(1,0)\left(\alpha_{12}+e^{i\theta}\,\alpha_{21}\right) & (16a) \\ H(1,0)\left(\alpha^*_{12}+e^{i\theta}\,\alpha^*_{21}\right). & (16b) \end{cases}$$

To fix θ, we note from Eq. (6) that, for any $a\in\mathbb{C}$, $|H(a,ae^{-i\theta})| = |H(ae^{-i\theta},a)|$, from which Eq. (16a) implies $|\cos\theta| = 1$. A similar procedure with Eq. (16b) yields the same result. Hence $e^{i\theta}=\pm 1$. Therefore,

$$H(\alpha_{12},\alpha_{21}) = \begin{cases} H(1,0)\,(\alpha_{12}\pm\alpha_{21}) & (17a) \\ H(1,0)\,(\alpha_{12}\pm\alpha_{21})^*. & (17b) \end{cases}$$

Since only the transition probability is of physical importance, only the modulus of $H(\alpha_{12},\alpha_{21})$ is relevant. Hence, without loss of generality, we can take $H(\alpha_{12},\alpha_{21}) = c\,(\alpha_{12}\pm\alpha_{21})$, with $c=|H(1,0)|$. Finally, Eq. (3) fixes $c=1$, leaving us with

$$H(\alpha_{12},\alpha_{21}) = \alpha_{12}\pm\alpha_{21}, \quad (18)$$

where the sign is the only remaining degree of freedom.

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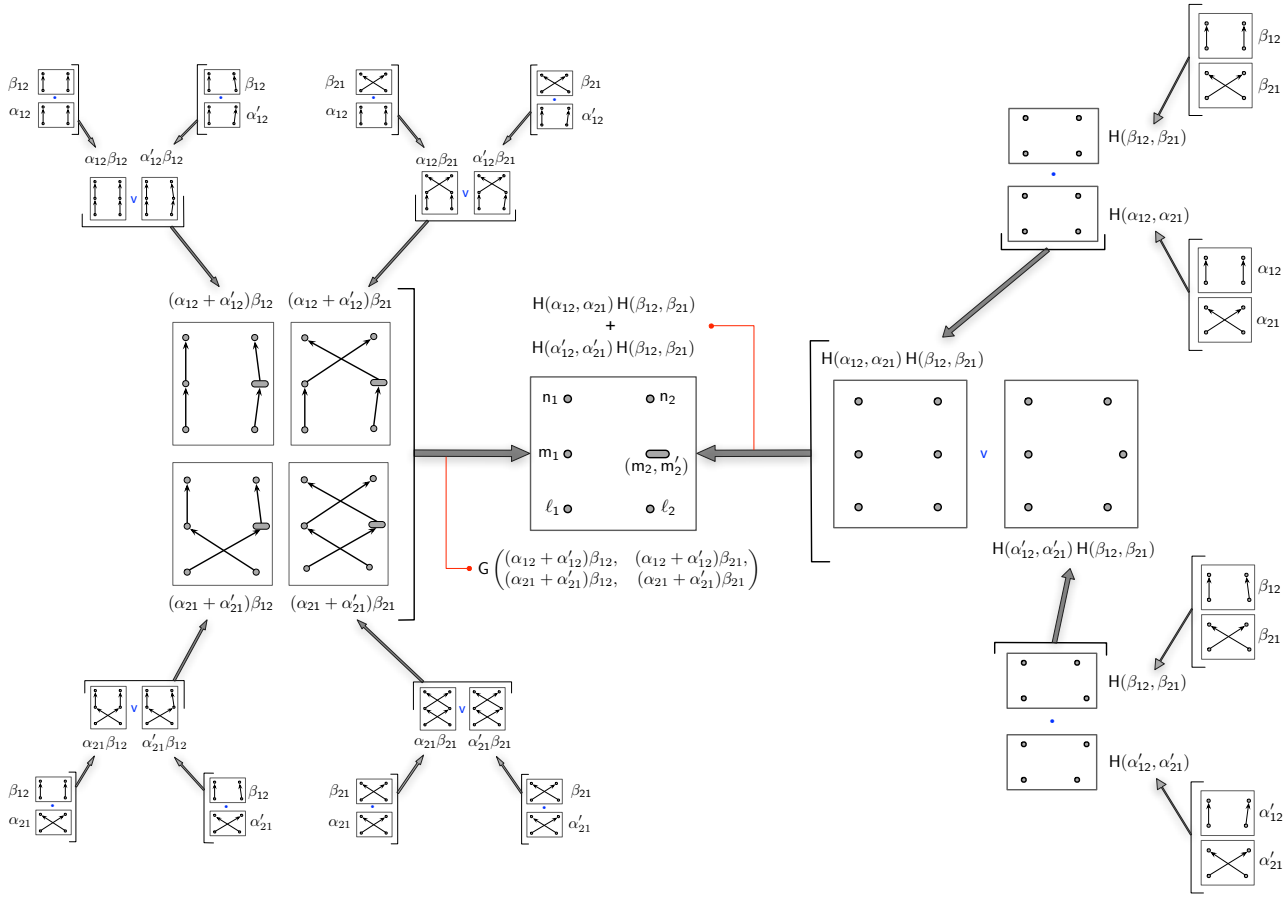

FIG. 5: In the center, a system of two indistinguishable particles is subject to measurements at three successive times. One of the outcomes of the measurements at t_2 is a coarse-graining of atomic outcomes m_2 and m'_2, denoted (m_2, m'_2). The amplitude of this indistinguishable-particle process can be computed in two different ways: (i) on the right, the process is decomposed into two processes combined in parallel ($\vee$), and each of these in turn is expressed as two processes combined in series ($\cdot$). The amplitudes of each of the four component processes can then be determined using H in terms of the amplitudes of distinguishable-particle transitions, yielding $[H(\alpha_{12},\alpha_{21}) + H(\alpha'_{12},\alpha'_{21})]\, H(\beta_{12},\beta_{21})$; (ii) on the left are shown the four compatible transitions of two distinguishable particles. The amplitudes of these transitions, $\gamma_{11} = (\alpha_{12}+\alpha'_{12})\beta_{12}, \gamma_{12} = (\alpha_{12}+\alpha'_{12})\beta_{21}, \gamma_{21} = (\alpha_{21}+\alpha'_{21})\beta_{12}$, and $\gamma_{22} = (\alpha_{21}+\alpha'_{21})\beta_{21}$, are computed using Feynman's sum and product rules as indicated. The amplitude of the indistinguishable-particle sequence is then obtained using the G-function to be $G(\gamma_{11},\gamma_{12},\gamma_{21},\gamma_{22})$.

Appendix A: System of Many Identical Subsystems

In this section, we present the derivation of the symmetrization postulate for N indistinguishable subsystems. This derivation closely follows the derivation for two indistinguishable subsystems given in the main text. For reasons of precision and clarity, we explicitly employ the the operational framework and notation used in Ref.[21]. For completeness, we first summarize the key features of this framework, and then describe the extensions we need in order to describe composite systems. We then summarize how one can describe Feynman paths and Feynman's amplitude rules in this operational framework. Having laid this foundation, we present the derivation itself.

1. Operational Framework

As explained in Refs.[21,30], the measurements and interactions which can be employed in a given experiment must be carefully circumscribed if they are to lead to a well-defined theoretical model. The overarching requirement is that, in the given experiment, the outcome probabilities of any measurement but the first are independent of any interactions with the system prior to the first measurement, a requirement we refer to as *experimental closure*. Intuitively, the first measurement 'screens off' the prior history of the system, rendering this prior history irrelevant insofar as making predictions about the outcomes of subsequent measurements in the experiment is concerned. The requirement of experimental closure naturally leads to the following definitions and constraints.

An experimental set-up is defined by specifying a source of physical systems, a sequence of measurements to be performed on a system on a run of the experiment, and the interactions with the system which occur between the measurements. In a run of an experiment, a physical system from the source passes through a sequence of measurements $\mathbf{L}, \mathbf{M}, \mathbf{N}, \ldots$, which respectively yield outcomes $\ell, m, n, \ldots$ at times $t_1, t_2, t_3, \ldots$. These outcomes are summarized in the *measurement sequence* $[\ell, m, n, \ldots]$. To avoid notational clutter, the measurements that yield these outcomes, the times at which these outcomes occur, and the nature of the system that is under examination, are all left implicit in the notation, and so must be inferred from the context. In between these measurements, the system may undergo interactions with the environment.

Over many runs of the experiment, the experimenter will observe the frequencies of the various possible measurement sequences, from which one can (using Bayes' rule) estimate the probability associated with each sequence. We define the probability $P(A)$ associated with sequence $A = [\ell, m, n, \ldots]$ as the probability of obtaining outcomes $m, n, \ldots$ conditional upon obtaining ℓ,

$$P(A) = \Pr(m, n, \ldots \mid \ell). \tag{A1}$$

A particular outcome of a measurement is either *atomic* or *coarse-grained*. An atomic outcome is one that cannot be more finely divided in the sense that the detector whose output corresponds to the outcome cannot be sub-divided into smaller detectors whose outputs correspond to two or more outcomes. An example of atomic outcomes are the two possible outcomes of a Stern-Gerlach measurement performed on a silver atom. A coarse-grained outcome is one that does not differentiate between two or more outcomes, an example being a Stern-Gerlach measurement where a detector's field of sensitivity encompasses the fields of sensitivity of two detectors each of which corresponds to a different atomic outcome. Abstractly, if measurement $\widetilde{\mathbf{M}}$ has an outcome which is a coarse-graining of the outcomes labeled 1 and 2 of measurement $\mathbf{M}$, the outcome of $\widetilde{\mathbf{M}}$ is labeled $(1, 2)$, and this notational convention naturally extends to coarse-graining of more than two atomic outcomes. In general, if all of the possible outcomes of a measurement are atomic, we shall call the measurement itself atomic. Otherwise, we say it is a coarse-grained measurement.

It is important that all of the measurements that are employed in an experimental set-up come from the same *measurement set*, $\mathcal{M}$, or are coarsened versions of measurements in this set. This ensures that all the measurements are probing the same aspect of the system. The set is operationally defined as follows. Measurement $\mathbf{M}$ forms a *measurement pair* with $\mathbf{L}$ if in both (a) experiment 1, where measurement $\mathbf{M}$ is performed on a given system immediately after it has been prepared using $\mathbf{L}$, and (b) experiment 2, where measurement $\mathbf{L}$ is used to prepare the given system, and measurement $\mathbf{M}$ is performed immediately afterwards, we have *experimental closure*, namely that the outcome probabilities of the final measurement in both instances are independent of interactions with the system prior to the preparation. The measurement set, $\mathcal{M}$, containing $\mathbf{M}$ is the set of all measurements that form a measurement pair with $\mathbf{M}$. Interactions that occur in the period of time *between* measurements are likewise selected from a set, $\mathcal{I}$, of possible interactions, which are such that they preserve closure when performed between any pair of measurements from $\mathcal{M}$. Experimental closure requires that the first measurement, $\mathbf{M}$, in an experiment is atomic; the remaining measurements must either lie in the same measurement set as $\mathbf{M}$ or be coarsened versions of measurements in this set.

a. Composite Systems

The operational framework sketched above is concerned with measurements that probe the entire system. But, in the case of a composite system, it is possible to perform measurements that only probe particular subsystems of the system. For example, consider a system subject to position measurements at successive times. If, at each of these times, two clicks are registered, we say that the system is a composite system of two particles.

More formally, consider an experimental arrangement in which, at each time, one measurement is performed on each of the two subsystems of a composite system. If, at time t_1, measurements $\mathbf{L_1}$ are performed on one subsystem and $\mathbf{L_2}$ on the other, and simultaneously yield outcomes ℓ_1 and ℓ_2, respectively, we shall notate the outcome of the two measurements as $(\ell_1 \colon \ell_2)$. If the measurement L_1 and L_2 are performed at time t_1, and measurements M_1 and M_2 are subsequently performed at t_2, the resulting measurement sequence can accordingly be written as $C = [(\ell_1 \colon \ell_2), (m_1 \colon m_2)]$. To ensure closure, the measurements $\mathbf{L_1}, \mathbf{L_2}, \mathbf{M_1}$, and $\mathbf{M_2}$ must all lie in the same measurement set.

If it is possible to carry out measurements immediately before t_1 and after t_2 whose outcomes together determine whether the subsystem detected at $\mathbf{M}_1$ originated at $\mathbf{L}_1$ or at $\mathbf{L}_2$, we say that these subsystems are distinguishable. Otherwise we say they are indistinguishable, in which case the statement that subsystem 1 originated at $\mathbf{L}_1$ and transitioned to, say, $\mathbf{M}_1$, is not, in general, operationally meaningful.

2. Operationalization of Feynman's paths and Feynman's rules

Consider an experimental set-up in which a physical system is subject to successive measurements $\mathbf{L}, \mathbf{M}, \mathbf{N}$ at successive times t_1, t_2, t_3, with there possibly being interactions with the system in the intervals between those measurements. Here and subsequently, we assume that the measurements and interactions in any such set-up are selected according to the constraints described above. We summarize the outcomes obtained in a given run of the experiment as the *sequence* $C = [\ell, m, n]$. This is the operational counterpart to a Feynman 'path'.

We now wish to formalize the idea that set-ups are interrelated in a particular ways. In Ref.[21], we considered three such relationships. First, the above set-up could be viewed as a *series concatenation* of two experiments, the first in which measurements $\mathbf{L}$ and $\mathbf{M}$ occur at times t_1 and t_2, yielding the sequence $A = [\ell, m]$, and the second in which measurements $\mathbf{M}$ and $\mathbf{N}$ occur at times t_2 and t_3, yielding $B = [m, n]$. In order to ensure that experimental closure is satisfied in the second experiment, measurement $\mathbf{M}$ must be atomic. Formally, we express this concatenation as

$$C = A \cdot B, \tag{A2}$$

where $\cdot$ is the *series* combination operator. More generally, the series operator can be used to concatenate two sequences provided their initial and final measurements are atomic, and the final measurement and outcome of the first sequence is the same as the initial measurement and outcome of the second sequence.

Second, one can consider a set-up which is identical to the one above, except that outcomes m and m' of $\mathbf{M}$ have been coarse-grained, so that one obtains the sequence $E = [\ell, (m, m'), n]$. Formally, we express the relationship of this sequence to the sequences $C = [\ell, m, n]$ and $D = [\ell, m', n]$ as

$$E = C \vee D, \tag{A3}$$

where $\vee$ is the *parallel* combination operation. More generally, the parallel operator can combine any two sequences which are identical except for differing in the outcome of a single measurement in the set-up, provided that this measurement is not the initial or final measurement.

a. Feynman's Rules in Operational Form

From the above definitions, it follows that the operators $\cdot$ and $\vee$ satisfy several symmetry relations, to which we collectively refer to as an *operational logic*. In Ref.[21], it is shown that Feynman's rules are the unique pair-valued representation of this logic consistent with a few additional assumptions. Writing $z(X)$ for the complex-valued *amplitude* that represents sequence X, one finds

$$z(A \vee B) = z(A) + z(B) \qquad \text{(amplitude sum rule)}$$

$$z(A \cdot B) = z(A) \cdot z(B) \qquad \text{(amplitude product rule)}$$

$$P(A) = |z(A)|^2 . \qquad \text{(probability rule)}$$

These are Feynman's rules for measurements on single quantum systems.

3. Derivation

The derivation for N indistinguishable subsystems closely follows the derivation for two indistinguishable subsystems given in the main text. The key postulates are as follows. Consider an experiment involving N indistinguishable subsystems, with measurements $L_1, L_2, \ldots, L_N$ performed at time t_1, and $M_1, M_2, \ldots, M_N$ performed at t_2. We postulate that the amplitude of the sequence $A = [\ell, m]$, where $\ell \equiv (\ell_1 \colon \ell_2 \colon \ldots \colon \ell_N)$ and $m \equiv (m_1 \colon m_2 \colon \ldots \colon m_N)$, is a function of the amplitudes of the transitions of N distinguishable particles compatible with A. There are $N!$ such transitions, each corresponding to some permutation, $\boldsymbol{\pi}$, in the symmetric group S_N: in a transition corresponding to $\boldsymbol{\pi}$, the subsystem that was measured at $\mathbf{L}_j$ is later found at $\mathbf{M}_{\boldsymbol{\pi}(j)}$, for $j = 1, 2, \ldots, N$. We notate this transition as $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}$, where $\boldsymbol{\pi}_1$ is the identity permutation here and subsequently. Accordingly, we write

$$z\,(A) = H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}), \tag{A4}$$

where the $\boldsymbol{\pi}_i$ are the $N!$ distinct permutations in S_N, and $\alpha_{\boldsymbol{\pi}}$ is the amplitude of the distinguishable-particle transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}$.

Similarly, for three stages of measurement, with each stage consisting of N measurements, we postulate that the sequence $C = [\ell, m, n]$, where $n \equiv (n_1 \colon n_2 \colon \ldots \colon n_N)$, has amplitude

$$z\,(C) = G(\boldsymbol{\Gamma}), \tag{A5}$$

where $\boldsymbol{\Gamma}_{ij} = \gamma_{ij}$ is the amplitude of the transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}_i \to \boldsymbol{\pi}_j$.

a. Labelling Invariance.

In an experimental situation involving N subsystems subject to measurement at two successive times, one can relabel the measurements $L_i \to L_{\pi'(i)}$ and $M_j \to M_{\pi(j)}$ for $i, j \in \{1, 2, \ldots, N\}$, where $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ are each permutations of N elements (see Fig. 4). The amplitudes computed for these two situations is given by $H(\alpha_{\boldsymbol{\pi}_1}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}})$ and $H(\alpha'_{\boldsymbol{\pi}_1}, \ldots, \alpha'_{\boldsymbol{\pi}_{N!}})$ respectively, where $\alpha'_{\boldsymbol{\pi}_i} = \alpha_{\boldsymbol{\pi}'^{-1}\boldsymbol{\pi}_i\boldsymbol{\pi}}$. Now, since the experimental situations are precisely the same, these amplitudes must yield the same transition probabilities. Hence, for any permutations $\boldsymbol{\pi}, \boldsymbol{\pi}'$,

$$|H(\alpha_{\boldsymbol{\pi}_1}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}})| = |H(\alpha'_{\boldsymbol{\pi}_1}, \ldots, \alpha'_{\boldsymbol{\pi}_{N!}})|, \tag{A6}$$

a condition we refer to as the *labelling invariance condition*.

In particular, setting $\alpha_{\boldsymbol{\pi}_i} = \delta_{i1}$ and $\boldsymbol{\pi}'$ equal to the identity permutation, we have $\alpha'_{\boldsymbol{\pi}_i} = \alpha_{\boldsymbol{\pi}_i\boldsymbol{\pi}} = \delta_{\boldsymbol{\pi}_i,\boldsymbol{\pi}}$. Define $Q(\boldsymbol{\pi}_i) \equiv H(0, \ldots, 0, 1, 0, \ldots, 0)$ where unity appears as the ith argument of H. Then,

$$|Q(\boldsymbol{\pi}_1)| = |Q(\boldsymbol{\pi})|,$$

so that, for all $\boldsymbol{\pi}$,

$$|Q(\boldsymbol{\pi})| = c, \tag{A7}$$

where $c \in \mathbb{R}_0^+$ is to be determined.

b. The G-product equation.

In the case where the outcomes at t_2 are atomic, we can compute the amplitude of the sequence $C = [\ell, m, n]$ in two different ways. First, writing $\alpha_{\boldsymbol{\pi}}$ as the amplitude of the distinguishable-particle transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}$ compatible with $[\ell, m]$ and similarly $\beta_{\boldsymbol{\pi}}$ as the amplitude of the transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}$ compatible with $[m, n]$, we can directly use Eq. (A5) to get

$$z(C) = G(\boldsymbol{\Gamma}),$$

where $\boldsymbol{\Gamma}_{ij} = \alpha_{\boldsymbol{\pi}_i}\,\beta_{\boldsymbol{\pi}_j}$ is the amplitude of the distinguishable-particle transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}_i \to \boldsymbol{\pi}_j$, computed using the amplitude product rule. Alternatively, we can write

$$C = [\ell, m] \cdot [m, n]$$

so that

$$z(C) = H(\alpha_{\boldsymbol{\pi}_1}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}})\, H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}).$$

Equating these two results, we obtain a functional equation, the G-product equation,

$$G(\boldsymbol{\Gamma}) = H(\alpha_{\boldsymbol{\pi}_1}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}})\, H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}). \tag{A8}$$

c. Coarse graining, and the additivity relation for H.

Next, consider an experiment with measurements at times t_1, t_2 and t_3 where measurement M_q, for some $q \in \{1, \ldots, N\}$, is coarse-grained, and has an outcome is $\tilde{m}_q = (m_q^{(1)}, m_q^{(2)})$. Suppose that the sequence $C = [\ell, m, n]$, where $m \equiv (m_1 \colon m_2 \colon \ldots \colon \tilde{m}_q \colon \ldots \colon m_N)$, is obtained. We can compute its amplitude in two ways.

Let $\alpha^{(r)}_{\boldsymbol{\pi}_i}$ be the amplitude of the transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}_i$ compatible with $[\ell, (m_1 \colon m_2 \colon \ldots \colon m_q^{(r)} \colon \ldots \colon m_N)]$, and likewise $\beta^{(r)}_{\boldsymbol{\pi}_j}$ the amplitude of the transition $\boldsymbol{\pi}_1 \to \boldsymbol{\pi}_j$ compatible with $[(m_1 \colon m_2 \colon \ldots \colon m_q^{(r)} \colon \ldots \colon m_N), n]$. We restrict attention to the special case where $\beta^{(r)}_{\boldsymbol{\pi}_j} = \beta_{\boldsymbol{\pi}_j}$. Then, we can directly apply Eq. (A5) to get

$$z(C) = G(\boldsymbol{\Gamma}),$$

where $\Gamma_{ij} = (\alpha^{(1)}_{\boldsymbol{\pi}_i} + \alpha^{(2)}_{\boldsymbol{\pi}_i})\beta_{\boldsymbol{\pi}_j}$ is computed using the amplitude sum and product rules. Using the G-product equation (Eq. (A8)), this can be written

$$z(C) = H(\alpha^{(1)}_{\boldsymbol{\pi}_1} + \alpha^{(2)}_{\boldsymbol{\pi}_1}, \ldots, \alpha^{(1)}_{\boldsymbol{\pi}_{N!}} + \alpha^{(2)}_{\boldsymbol{\pi}_{N!}})\, H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}).$$

Alternatively, we can write the sequence C as

$$C = \bigvee_{r=1}^{2} [\ell, (m_1 \colon m_2 \colon \ldots \colon m_q^{(r)} \colon \ldots \colon m_N), n]$$

whose amplitude is given using the amplitude sum rule by

$$z(C) = \sum_{r=1}^{2} H(\alpha^{(r)}_{\boldsymbol{\pi}_1}, \alpha^{(r)}_{\boldsymbol{\pi}_2}, \ldots, \alpha^{(r)}_{\boldsymbol{\pi}_{N!}})\, H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}).$$

If $H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}) \neq 0$ for some values of $\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}$, we can equate these two expressions for $z(C)$ to obtain the additivity relation

$$H(\alpha^{(1)}_{\boldsymbol{\pi}_1} + \alpha^{(2)}_{\boldsymbol{\pi}_1}, \alpha^{(1)}_{\boldsymbol{\pi}_2} + \alpha^{(2)}_{\boldsymbol{\pi}_2}, \ldots, \alpha^{(1)}_{\boldsymbol{\pi}_{N!}} + \alpha^{(2)}_{\boldsymbol{\pi}_{N!}}) = \sum_{r=1}^{2} H(\alpha^{(r)}_{\boldsymbol{\pi}_1}, \alpha^{(r)}_{\boldsymbol{\pi}_2}, \ldots, \alpha^{(r)}_{\boldsymbol{\pi}_{N!}}). \tag{A9}$$

The case $H(\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}) = 0$ for all $\beta_{\boldsymbol{\pi}_1}, \ldots, \beta_{\boldsymbol{\pi}_{N!}}$ is the trivial solution that implies that every transition of N indistinguishable particles has zero probability. This solution can therefore be discarded.

d. Simplification of expression for H.

From the additivity relation Eq. (A9), it follows that

$$H(\alpha_{\boldsymbol{\pi}_1} + 0, \alpha_{\boldsymbol{\pi}_2} + 0, \ldots, \alpha_{\boldsymbol{\pi}_{N!-1}} + 0, 0 + \alpha_{\boldsymbol{\pi}_{N!}}) = H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!-1}}, 0) + H(0, 0, \ldots, 0, \alpha_{\boldsymbol{\pi}_{N!}}).$$

Iteratively applying the additivity relation, we obtain

$$\begin{aligned} H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) &= H(\alpha_{\boldsymbol{\pi}_1}, 0, \ldots, 0) + H(0, \alpha_{\boldsymbol{\pi}_2}, 0, \ldots, 0) + \cdots + H(0, 0, \ldots, 0, \alpha_{\boldsymbol{\pi}_{N!}}) \\ &= J_1(\alpha_{\boldsymbol{\pi}_1}) + J_2(\alpha_{\boldsymbol{\pi}_2}) + \cdots + J_{N!}(\alpha_{\boldsymbol{\pi}_{N!}}), \end{aligned} \tag{A10}$$

where we have defined $J_i(z) \equiv H(0, \ldots, 0, z, 0, \ldots, 0)$, where z appears as the ith argument of H.

Writing $J(z) \equiv J_1(z)$, Eq. (A8) implies

$$J(z_1 z_2)\, J(1) = J(z_1)\, J(z_2). \tag{A11}$$

If $J(1) = 0$, the above equation implies that $J(z) = 0$ for all z, and Eqs. (A7) and (A10) then lead to the trivial solution $H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) = 0$. Hence $J(1) \neq 0$.

We can interrelate the $J_i(\cdot)$ using the G-product equation (Eq. (A8)). Taking $\Gamma_{ij} = \delta_{ik}\delta_{j\ell} z$, the G-product equation implies

$$G(\boldsymbol{\Gamma}) = J_k(z)\, J_\ell(1) = J_k(1)\, J_\ell(z).$$

Hence, Eq. (A10) can be written solely in terms of $J(z)$ and the constants $J_i(1)$:

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) = \frac{1}{J(1)} \sum_{i=1}^{N!} J_i(1) J(\alpha_{\boldsymbol{\pi}_i}).$$

To fix the form of $J(z)$, we note that, from Eq. (A9),

$$J(z_1 + z_2) = J(z_1) + J(z_2). \tag{A12}$$

Writing $J(z) = J(1)\, f(z)$ transforms Eqs. (A11) and (A12) into Eqs. (15), whose continuous solutions are $f(z) = z$, $f(z) = z^*$ or $f(z) = 0$. The zero solution is inadmissible since, from its definition, $f(1) = 1$. Therefore,

$$J(z) = \begin{cases} J(1)\, z & \text{(A13a)} \\ J(1)\, z^*, & \text{(A13b)} \end{cases}$$

so that, writing $J_i(1)$ as $Q(\boldsymbol{\pi}_i)$,

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) = \begin{cases} \displaystyle\sum_{\boldsymbol{\pi} \in S_N} Q(\boldsymbol{\pi})\, \alpha_{\boldsymbol{\pi}} & \text{(A14a)} \\ \displaystyle\sum_{\boldsymbol{\pi} \in S_N} Q(\boldsymbol{\pi})\, \alpha^*_{\boldsymbol{\pi}}. & \text{(A14b)} \end{cases}$$

a. Determination of values of the $Q(\boldsymbol{\pi})$. To determine the values of $Q(\boldsymbol{\pi})$ for all possible permutations $\boldsymbol{\pi} \in S_n$, we first show that $Q(\boldsymbol{\pi}) = \pm c$, and then show that $Q(\boldsymbol{\pi}) = Q(\boldsymbol{\pi}')$ whenever $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ are both odd or both even.

i. Establishing that $Q(\boldsymbol{\pi}) = \pm c$. Using Eq. (A7), we see that $Q(\boldsymbol{\pi})$ can be written as $Q(\boldsymbol{\pi}_1)\, e^{i\theta_{\boldsymbol{\pi}}}$, where $\theta_{\boldsymbol{\pi}}$ is an angle to be determined. Hence, we can rewrite Eqs. (A14) as

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \ldots, \alpha_{\boldsymbol{\pi}_{N!}}) = \begin{cases} Q(\boldsymbol{\pi}_1) \displaystyle\sum_{\boldsymbol{\pi} \in S_N} e^{i\theta_{\boldsymbol{\pi}}}\, \alpha_{\boldsymbol{\pi}} & \text{(A15a)} \\ Q(\boldsymbol{\pi}_1) \displaystyle\sum_{\boldsymbol{\pi} \in S_N} e^{i\theta_{\boldsymbol{\pi}}}\, \alpha^*_{\boldsymbol{\pi}}. & \text{(A15b)} \end{cases}$$

Now, using Eq. (A6), setting $\alpha_{\boldsymbol{\pi}_1} = a, \alpha_{\boldsymbol{\pi}} = a e^{-i\theta_{\boldsymbol{\pi}}}$ and all others zero, and $\alpha'_{\boldsymbol{\pi}_1} = \alpha_{\boldsymbol{\pi}}, \alpha'_{\boldsymbol{\pi}} = \alpha_{\boldsymbol{\pi}_1}$, and all others zero, with $a \in \mathbb{C}$, Eq. (A15a) gives

$$|2a\, Q(\boldsymbol{\pi}_1)| = |2a\, Q(\boldsymbol{\pi}_1) \cos\theta_{\boldsymbol{\pi}}|,$$

which implies $\cos\theta_{\boldsymbol{\pi}} = \pm 1$. A similar procedure with Eq. (A15b), except setting $\alpha_{\boldsymbol{\pi}} = a e^{i\theta_{\boldsymbol{\pi}}}$, yields the same result. Therefore,

$$Q(\boldsymbol{\pi}) = \pm c. \tag{A16}$$

ii. Establishing $Q(\boldsymbol{\pi}) = Q(\boldsymbol{\pi}')$ whenever $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ are both odd or both even. Let $\alpha_{\boldsymbol{\pi}_i} = k(\delta_{\boldsymbol{\pi}_1, \boldsymbol{\pi}_i} + \delta_{\boldsymbol{\rho}, \boldsymbol{\pi}_i})$, where $\boldsymbol{\rho} \neq \boldsymbol{\pi}_1$ is some permutation, and k is some constant. Then the labeling invariance condition, Eq. (A6), with $\boldsymbol{\pi}' = \boldsymbol{\pi}$ implies that, for any $\boldsymbol{\pi}$,

$$|Q(\boldsymbol{\pi}_1) + Q(\boldsymbol{\rho})| = |Q(\boldsymbol{\pi}_1) + Q(\boldsymbol{\pi}^{-1}\, \boldsymbol{\rho}\, \boldsymbol{\pi})|.$$

where we have made use of Eqs. (A14). Therefore, due to Eq. (A16), for any $\boldsymbol{\pi}$,

$$Q(\boldsymbol{\rho}) = Q(\boldsymbol{\pi}^{-1}\, \boldsymbol{\rho}\, \boldsymbol{\pi}).$$

That is, any two permutations in the same conjugacy class have the same Q-value.

Next, again let $\alpha_{\boldsymbol{\pi}_i} = k(\delta_{\boldsymbol{\pi}_1, \boldsymbol{\pi}_i} + \delta_{\boldsymbol{\rho}, \boldsymbol{\pi}_i})$, but, in Eq. (A6), take $\boldsymbol{\pi}'$ to be the identity permutation. Then, for any $\boldsymbol{\pi}$,

$$|Q(\boldsymbol{\pi}_1) + Q(\boldsymbol{\rho})| = |Q(\boldsymbol{\pi}) + Q(\boldsymbol{\rho}\boldsymbol{\pi})|. \tag{A17}$$

In particular, for any $\boldsymbol{\pi} \in \mathrm{Cl}(\boldsymbol{\rho})$, in which case we just have established that $Q(\boldsymbol{\rho}) = Q(\boldsymbol{\pi})$, again making use of Eqs. (A14) and (A16),

$$Q(\boldsymbol{\pi}_1) = Q(\boldsymbol{\rho}\boldsymbol{\pi}).$$

That is, the product of any pair of permutations in the same conjugacy class has the same Q-value as the identity permutation. In particular, this holds true for the product $\boldsymbol{\omega} = \boldsymbol{\tau}\boldsymbol{\tau}'$ of any two transpositions $\boldsymbol{\tau}, \boldsymbol{\tau}'$ since $\mathrm{Cl}(\boldsymbol{\tau}) = \mathrm{Cl}(\boldsymbol{\tau}')$.

To show that all products of an even number of transpositions have the same Q-value as the identity, we proceed by induction. Let P_k be the proposition that $Q(\boldsymbol{\pi}_1) = Q(\boldsymbol{\omega}_1 \dots \boldsymbol{\omega}_k)$ for any $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_k$ where $\boldsymbol{\omega}_i$ is a product of two transpositions. We have already established that P_1 is true. Suppose, now, that P_k is true for some $k > 1$. Then, setting $\boldsymbol{\rho} = \boldsymbol{\omega}_1$ and $\boldsymbol{\pi} = \boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1}$ in Eq. (A17), we obtain

$$|Q(\boldsymbol{\pi}_1) + Q(\boldsymbol{\omega}_1)| = |Q(\boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1}) + Q(\boldsymbol{\omega}_1\boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1})|\,.$$

Now, P_k implies $Q(\boldsymbol{\pi}_1) = Q(\boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1})$. Therefore,

$$Q(\boldsymbol{\omega}_1) = Q(\boldsymbol{\omega}_1\boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1}).$$

But we have already established that $Q(\boldsymbol{\pi}_1) = Q(\boldsymbol{\omega}_1)$ for any $\boldsymbol{\omega}_1$. Therefore, for all $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{k+1}$,

$$Q(\boldsymbol{\pi}_1) = Q(\boldsymbol{\omega}_1\boldsymbol{\omega}_2 \dots \boldsymbol{\omega}_{k+1}),$$

which is P_{k+1}. Therefore, by induction, P_k is true for all k. But every even permutation can be written as a product of an even number of transpositions. Therefore, for any even permutation matrix $\boldsymbol{\pi}$,

$$Q(\boldsymbol{\pi}) = Q(\boldsymbol{\pi}_1). \tag{A18}$$

Finally, in Eq. (A17), take $\boldsymbol{\pi}$ equal to any even permutation, and $\boldsymbol{\rho}$ any odd permutation. Then, due to Eq. (A18), it follows that, for any odd $\boldsymbol{\rho}$ and any even $\boldsymbol{\pi}$,

$$Q(\boldsymbol{\rho}) = Q(\boldsymbol{\rho}\boldsymbol{\pi}).$$

But any odd permutation can be written as the product of any given odd permutation and some even permutation. Therefore, all odd permutations have the same Q-value. With these results for $Q(\boldsymbol{\pi})$, Eq. (A14) becomes

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \dots, \alpha_{\boldsymbol{\pi}_{N!}}) = \begin{cases} \displaystyle\sum_{\boldsymbol{\pi} \in S_N} K\left(\mathrm{sgn}(\boldsymbol{\pi})\right) \alpha_{\boldsymbol{\pi}} & \text{(A19a)} \\ \displaystyle\sum_{\boldsymbol{\pi} \in S_N} K\left(\mathrm{sgn}(\boldsymbol{\pi})\right) \alpha^*_{\boldsymbol{\pi}}, & \text{(A19b)} \end{cases}$$

where $K(\cdot) = \pm c$, and $\mathrm{sgn}(\boldsymbol{\pi})$ takes the value $+1$ or -1 according to whether $\boldsymbol{\pi}$ is even or odd.

Insofar as the probability of the transition of the system of N indistinguishable particles is concerned, the overall sign of H and the complex conjugation are irrelevant. More generally, consider a system, $\mathcal{S}$, that consists of subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ consists of N indistinguishable particles of one type (say, electrons). Suppose, first, that $\mathcal{S}_2$ only contains particles that can be distinguished from those in $\mathcal{S}_1$. As described above, let measurements $\mathbf{L}_1, \dots, \mathbf{L}_N$ and $\mathbf{M}_1, \dots, \mathbf{M}_N$ be performed on $\mathcal{S}_1$ at times t_1 and t_2. Additionally, let measurements $\mathbf{U}$ and $\mathbf{V}$ be performed on $\mathcal{S}_2$ at t_1 and t_2, respectively, yielding outcomes u and v. Let $\tilde{\alpha}_{\boldsymbol{\pi}}$ be the amplitude of the transition of $\mathcal{S}$ from $(\ell\colon u)$ to $(m\colon v)$ in which the particles in $\mathcal{S}_1$ are treated as distinguishable and make the transition described by $\boldsymbol{\pi}$. Then, by the same argument as described above, the amplitude of the process from $(\ell\colon u)$ to $(m\colon v)$ where the particles in $\mathcal{S}_1$ are treated as indistinguishable is given by $H(\tilde{\alpha}_{\boldsymbol{\pi}_1}, \tilde{\alpha}_{\boldsymbol{\pi}_2}, \dots, \tilde{\alpha}_{\boldsymbol{\pi}_{N!}})$, and the transition probability is again unaffected by the overall sign or complex conjugation of H. In the case that $\mathcal{S}_2$ *does* contain, say, M particles that are indistinguishable from those in $\mathcal{S}_1$, the boundaries of $\mathcal{S}_1$ must be redrawn to encompass them. The resulting situation, namely a system composed of subsystem $\mathcal{S}'_1$, containing $N+M$ indistinguishable particles, and subsystem $\mathcal{S}'_2$, containing only particles that are distinguishable from those in $\mathcal{S}'_1$, is of the same type the as one previously considered.

Therefore, in general, the overall sign of H and the complex conjugation are irrelevant insofar as predictions are concerned, and can be discarded without any loss of generality. Finally, the requirement that $|H(1, 0, \dots, 0)| = 1$ fixes $c = 1$, leaving us with

$$H(\alpha_{\boldsymbol{\pi}_1}, \alpha_{\boldsymbol{\pi}_2}, \dots, \alpha_{\boldsymbol{\pi}_{N!}}) = \sum_{\boldsymbol{\pi} \in S_N} \left(\mathrm{sgn}(\boldsymbol{\pi})\right)^{\sigma} \alpha_{\boldsymbol{\pi}}, \tag{A20}$$

where $\sigma = 0$ or $\sigma = 1$ is the only remaining degree of freedom, corresponding respectively to bosons and fermions.

Appendix B: Solution of a pair of functional equations.

We solve Eqs. (15a) and (15b) with the aid of one of Cauchy's standard functional equations,

$$h(x_1 + x_2) = h(x_1) + h(x_2), \tag{B1}$$

where h is a real function and $x_1, x_2 \in \mathbb{R}$. Its continuous solution[31] is $h(x) = ax$ with $a \in \mathbb{R}$.

Setting $z_1 + z_2 = x + iy$, with $x, y \in \mathbb{R}$, in Eq. (15a) gives

$$f(x + iy) = f(x) + f(iy).$$

Applying Eq. (15a) again on $f(x_1 + x_2)$ and $f(iy_1 + iy_2)$ then implies

$$\begin{aligned} f(x_1 + x_2) &= f(x_1) + f(x_2) \\ f\,(iy_1 + iy_2) &= f(iy_1) + f(iy_2). \end{aligned}$$

The real and imaginary parts of both of these equations each have the form of Eq. (B1), and therefore have solutions

$$f(x) = \alpha x \qquad \text{and} \qquad f(iy) = \beta y$$

with $\alpha, \beta \in \mathbb{C}$, so that

$$f(x + iy) = \alpha x + \beta y. \tag{B2}$$

From Eq. (15b),

$$f(1 \cdot 1) = f(1)f(1) \qquad \text{and} \qquad f(i \cdot i) = f(i)f(i),$$

which, due to Eq. (B2), imply

$$\alpha = \alpha^2 \qquad \text{and} \qquad -\alpha = \beta^2.$$

These have solutions $(\alpha, \beta) = (0, 0), (1, i)$ and $(1, -i)$, which correspond to $f(z) = 0$, $f(z) = z$ and $f(z) = z^*$.